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REPORT NO.
ORDBB-TK-432

EQUATIONS OF MOTION FOR MISSILES
WITH
SIX DEGREES OF FREEDOM

JANUARY 1960

ATOMIC WEAPONS LABORATORIES
RESEARCH AND DEVELOPMENT

FELTMAN RESEARCH & ENGINEERING LABORATORIES

PICATINNY ARSENAL, DOVER, N. J.

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EQUATIONS OF MOTION FOR MISSILES WITH SIX DEGREES OF FREEDOM

PART 1

FORCES AND TORQUES ACTING ON THE MISSILE

JANUARY 1960

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Introduction

The literature on this subject is extensive and diverse in treatment of special cases, choice of reference coordinates, degrees of approximations, forms of solution, symbols, and nomenclature. One feature common to almost all references studied is the failure to emphasize the physical factors underlying the numerous aerodynamic coefficients. The result is that the relative importance of these factors has received little consideration. This report is prepared primarily for the benefit of those who are not experts in this field, but who must employ computing machines for obtaining numerical solutions to the equations submitted by experts. The differential equations for the general case will be derived from basic mechanical principles carrying as far as possible the physical factors before introducing the complicated dependencies of the numerous aerodynamic coefficients.

Insolvability of General Problem

It must be realized that the general problem, taking into account all aerodynamic factors, is incapable of solution by analytic means. The problem is insolvable for several reasons: (a) exact physical laws governing the aerodynamic forces have not yet been found; hence, empirical relations have to be used, (b) the aerodynamic forces depend in a complicated way upon the characteristics of the air, the velocity, the size, and the shape of the missile; simplifying assumptions and approximations for each special case have thus been necessary to describe the motion in special cases, and (c) no analytic solutions have yet been found for the nonlinear differential equations describing this type of motion.

In spite of these handicaps, the conventional differential equations of motion containing all available information have been solved by using one or more approximations, depending on the nature of each particular special case. By numerical techniques, the motions of missiles are being predicted with a high degree of success, from a practical point of view. In this report the attempt will be made to describe the origin of each force and torque acting on a missile, to provide a basis for judging the validities of approximations made.

Mechanical Factors

Attention is confined to a long, symmetrical missile (e.g., body of revolution) which may or may not be spinning as it moves through the air. The effects of gravity, wind, and the earth's curvature and rotation (Coriolis forces) will be excluded. These can be introduced later into the equations of motion,

independent of the aerodynamic forces. In general, the axis of the missile will make an angle Θ, the yaw angle, with the velocity relative to the air, V. The plane formed by these two lines, which will be called the yaw plane, may rotate about V, as shown in Figure 1. This produces combined pitching and yawing, or complex yaw.

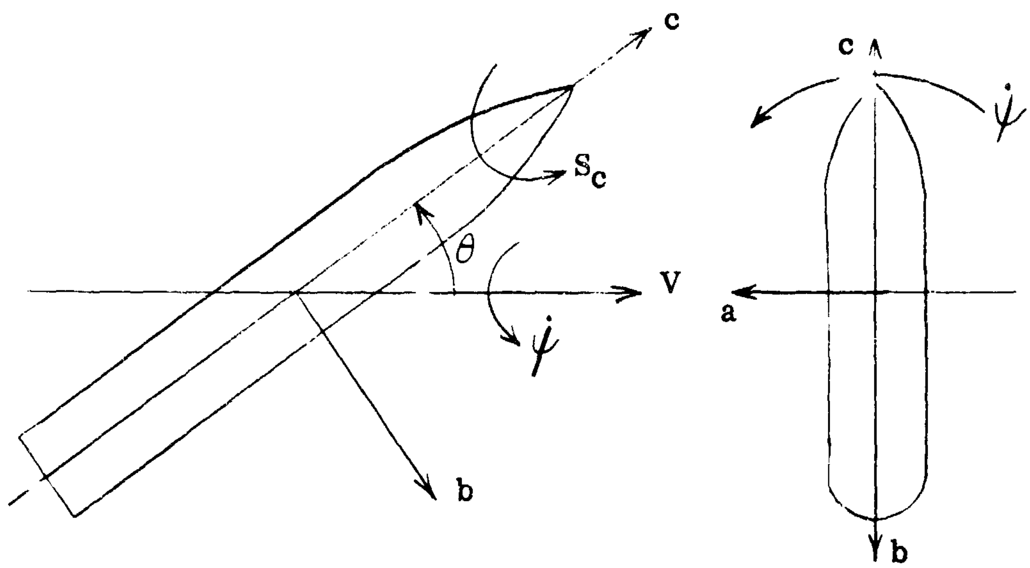

Figure 1

Motion of missile relative to air, showing yaw angle Θ.

The missile axes a', b' (not shown in the figure) and c are the principal axes of inertia; ; A = B > C are the moments of inertia, C being about the axis of revolution. These axes are rigidly attached to the missile and spin with it (if it has spin). An attempt is made in this presentation to adopt symbols for axes of reference that will require only a few subscripts for identification of force, torque and spin components. The spin of the missile about its axis will be designated by S_c.

The description of the effects of forces and torques on the missile is most naturally described by employing the orthogonal axes shown in Figure 1. Axis b lies in the yaw plane, and a is the line of nodes, the moving intersection of the (a b) plane with a reference plane perpendicular to V. The transformation to other reference triads will be made later, when the most convenient coordinates for particular problems will have to be chosen.

Forces and Torques

The model shown in Figure 2 was built to illustrate the forces and torques acting on a missile. The long cylinder mounted in gimbals represents the missile. Red plugs are attached to represent the forces and blue plugs the torques. The coordinates for the missile framework are the (abc) triad. The spinning missile itself is represented by the first collar at the lower end, bearing the axes a' and b' which can be rotated about the axis c to represent the spin of the missile. Rotatable on this is another collar bearing the jet factors, which rotate relative to the missile triad (a'b'c).

Forces will be derived for missiles having increasingly complex motions. In purely forward motion at zero yaw angle the resultant aerodynamic force is the axial drag which opposes the motion and points backward. If the missile is spinning, there will also be a spin decelerating torque T_A directed backwards.

In the case of "cross velocity", a motion in which the missile moves obliquely at constant yaw angle, the resultant force is resolved into two components, F_A and F_N, parallel and perpendicular, respectively, to the missile axis. In some treatments the resolution is parallel and perpendicular to V, as in airplane studies, where the plane flies nearly level, and where these two components are called "drag" and "lift". These names have been carried over into the general treatment. The word "drag" still has the same mechanical significance, i. e., a resistance to forward motion. "Lift" is no longer confined to a vertically upward force; it may act horizontally or even downward in some missile orientations.

The force F_N, called "normal force" or "cross force due to cross velocity" may be derived from a consideration of the motion of air relative to the missile, as shown in Figure 3, e. g., for a smooth artillery shell with zero spin.

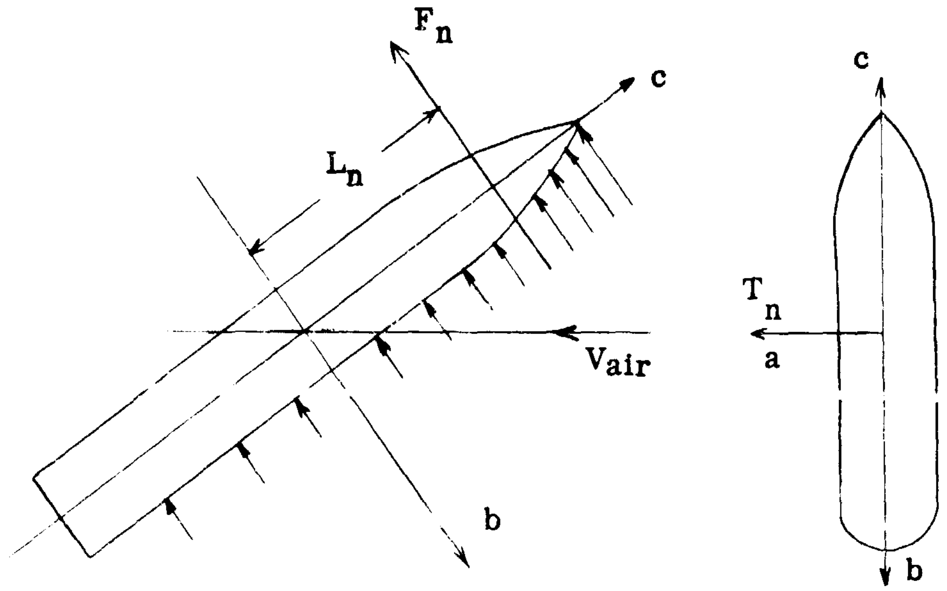

Figure 3

Normal Forces Acting on Missile

The vectors along the missile represent the forces due to cross velocity of the air against the shell. Their resultant is the normal force F_N, which in this case acts at a distance L_N forward of the center of gravity. It is inconvenient for analysis when the line of action of a force does not pass through the center of gravity; therefore, the line of action is transferred to the center of gravity, employing the standard procedure found in mechanics: A null pair (F_N and $-F_N$) is applied at the center of gravity. The original F_N is then combined with $-F_N$ to form a couple whose moment is $L_N F_N$ leaving a new F_N acting at the center of gravity. This is the procedure for deriving all the other torques in this problem and need not be described again. Because of the difficulty of measuring the lever arm L for a force, the practice in aeroballistics is to measure the corresponding torque in a wind tunnel, assigning to it a separate coefficient K. Strictly speaking this experimental value of torque should equal LF.

The aerodynamic forces and torques being considered are collected in Table 1. All are functions of air density and speed of sound, velocity V, yaw angle Θ, and the shape and size of the missile. These functional dependencies are tacitly understood during the derivation of the equations. They will be brought into consideration later, when it becomes necessary to take them into account.

Subscripts attached to the force symbols will be those conventionally adopted for the ballistic coefficients, just to relate our simplified symbols with existing usage. The subscript N refers to effects of cross velocity, and S to cross spin. Torques, contrary to convention, are given the same subscript as the corresponding force.

The complex notation adopted by most writers on this subject needs some explanation. The cross velocity is usually denoted by $\xi = V_{a'} + V_{b'}$, where a' and b' are any two perpendicular axes. In the present choice of axes the cross velocity lies entirely in the yaw plane, hence ξ lies in this plane and becomes equal to V_b.

The "cross spin", which is considered next, is represented by $\eta = S_{a'} + iS_{b'}$; however; unlike ξ, it may have any direction, depending on the cross spin components, and can be confined to neither the node line nor the yaw plane. Hence, the combined effects of both components S_a and S_b must be included in this analysis.

Cross spin S_a about the node line produces forces against the missile which are perpendicular to the axis. These forces, shown in Figure 4, decrease from a maximum at either end to zero at the center of gravity, having different senses at opposite ends. The resultant F_{sb} may be either fore or aft. The torque will be represented by a vector T_a in the direction a, in a sense opposing the spin S_a.

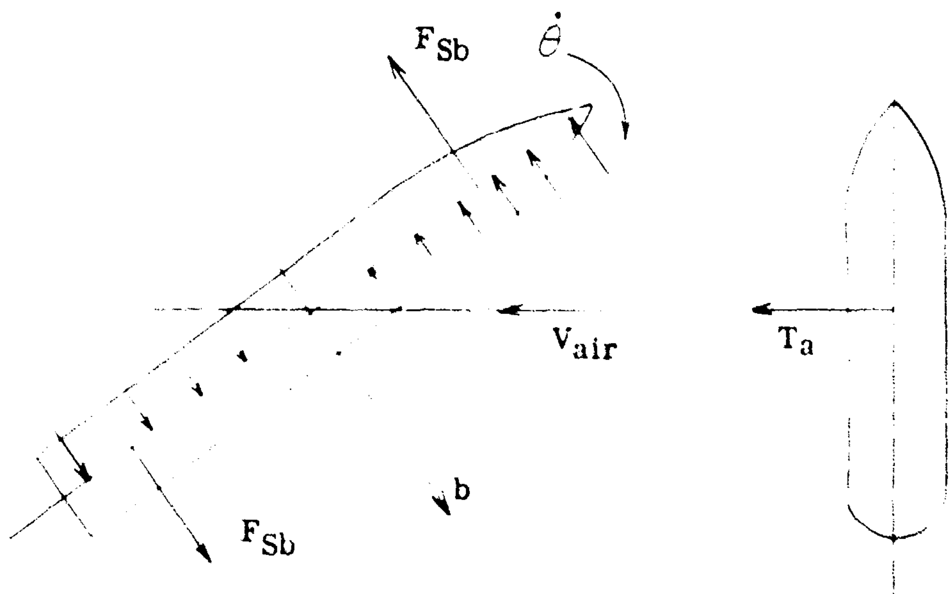

Figure 4

Force Due to Cross Spin

A spin about the b axis will produce a similar force and torque, represented by F_{Sa} and T_{Sb}

In general, the cross spin will be about some intermediate axis, and its effects will be the vector sums $F_{Sa} + F_{Sb}$ and $T_{Sa} + T_{Sb}$. In any case the torque will oppose the change in Θ, and is called the damping torque.

Magnus Factors

If the missile spins about its axis, the cross velocity and cross spin motions produce Magnus forces and torques. The additional subscript U is used to distinquish them from the factors previously derived.

Consider again the cross velocity, where the missile is now spinning about its axis. In Figure 5, the velocity of air against the missile is shown as V_N (normal to the axis c) The front view shows the air entrained by the spinning

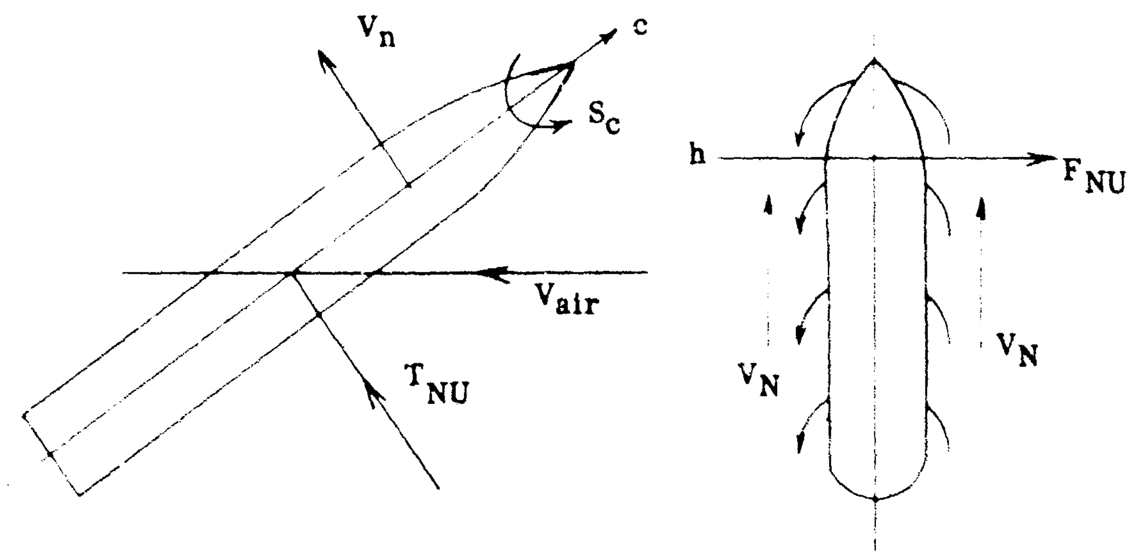

Figure 5

Magnus Factors are to Cross Velocity

missile (the boundary layer) moving counter to V_N on the left side, producing a higher pressure at h than on the other side. The pressures on the two sides

TABLE I

FORCES & TORQUES

NAME & DESCRIPTION / AXES	FORCES: a (NODE LINE)	FORCES: b	FORCES: c	TORQUES: a (NODE LINE)	TORQUES: b	TORQUES: c
AXIAL DRAG; SPIN DECELERATING TORQUE			$F_A = -\rho d^2 v^2 K_{DA}$			$T_A = -\rho d^4 v s_c K_A$
CROSS FACTORS DUE TO CROSS VELOCITY: NORMAL FORCE; OVERTURNING OR RESTORING TORQUE		$F_N = -\rho d^2 v_b v K_N$		$T_N = \pm\rho d^3 v_b v K_M$		
CROSS SPIN: CROSS FORCE; DAMPING TORQUE	$F_{Sa} = \mp\rho d^3 v s_b K_S$	$F_{Sb} = \pm\rho d^3 v s_a K_S$		$T_{Sa} = -\rho d^4 v s_a K_H$	$T_{Sb} = -\rho d^4 v s_b K_H$	
MAGNUS FACTORS DUE TO CROSS VELOCITY	$F_{NU} = -\rho d^3 v_b s_c K_F$				$T_{NU} = \mp\rho d^4 v_b s_c K_T$	
CROSS SPIN	$F_{SUa} = \pm\rho d^4 s_a s_c K_{XF}$	$F_{SUb} = \mp\rho d^4 s_b s_c K_{XF}$		$T_{SUa} = -\rho d^5 s_b s_c K_{XT}$	$T_{SUb} = -\rho d^5 s_a s_c K_{XT}$	
LAG FACTORS DUE TO ACCELERATIONS (SEE BRL 858)	$F_{LSa} = \pm\rho d^4 \dot{s}_b K_{LS}$ $F_{LNU} = \rho d^4 \dot{v}_b s_c K_{LF}$ $F_{LSUa} = \pm\rho d^5 \frac{s_c}{v} \dot{s}_a K_{LXF}$	$F_{LN} = \rho d^3 \dot{v}_b v K_{LN}$ $F_{LSb} = \pm\rho d^4 \dot{s}_a K_{LS}$ $F_{LSUb} = \pm\rho d^5 \frac{s_c}{v} \dot{s}_b K_{LXF}$		$T_{LN} = \mp\rho d^4 \dot{v}_b v K_{LM}$ $T_{LSa} = \rho d^5 v \dot{s}_a K_{LH}$ $T_{LSUa} = -\rho d^6 \frac{s_c}{v} \dot{s}_b K_{LXT}$	$T_{LSb} = \rho d^5 v \dot{s}_b K_{LH}$ $T_{LNU} = \pm\rho d^5 \dot{v}_b s_c K_{LT}$ $T_{LSUb} = -\rho d^6 \frac{s_c}{v} \dot{s}_a K_{LXT}$	
FACTORS DUE TO MISSILE ASYMMETRY δ_ϵ = EFFECTIVE ASYMMETRY ANGLE (ϵ FOR ECCENTRICITY)	$F_{\epsilon a} = \rho d^2 v^2 \delta_\epsilon K_{N\epsilon} \cos ts_c$	$F_{\epsilon b} = \rho d^2 v^2 \delta_\epsilon K_{N\epsilon} \sin ts_c$		$T_{\epsilon a} = \rho d^3 v^2 \delta_\epsilon K_{M\epsilon} \cos ts_c$	$T_{\epsilon b} = \rho d^3 v^2 \delta_\epsilon K_{M\epsilon} \sin ts_c$	
JET FACTORS: (ROTATE WITHIN A' B') F_J AND T_J AT ANGLE h	$F_{Ja} = F_J \cos ts_c$	$F_{Jb} = F_J \sin ts_c$	F_{Jc} = AXIAL COMPONENT	$T_{Ja} = T_J \cos (ts_c - h)$	$T_{Jb} = T_J \sin (ts_c - h)$	T_{Jc} = AXIAL COMPONENT
GUIDANCE FACTORS (ROTATE WITH A' B')	F_{Ga}	F_{Gb}	F_{Gc} = AXIAL COMPONENT	T_{Ga}	T_{Gb}	T_{Gc} = AXIAL COMPONENT

THE APPEARANCE OF V INSTEAD OF V_c IN THE ABOVE FORMULA MAY BE OPEN TO QUESTION, DEPENDING UPON DEFINITION OF BALLISTIC COEFFICIENTS.

SYMBOLS:

abc ARE REFERENCE AXES SHOWN IN FIGURE I

ABC ARE MOMENTS OF INERTIA ABOUT MISSILE AXES a'b'c

θ = YAW ANGLE, BETWEEN V AND V_c

ψ = ANGULAR VELOCITY OF YAW PLANE ABOUT V

VECTOR RELATIONS:

$V_a + V_b + V_c = V$ = VELOCITY RELATIVE TO AIR

$S_a + S_b + S_c = S$ = TOTAL SPIN

of the missile are thus unbalanced, producing a net force F_{Nu} in the negative sense along the a axis. This is the "Magnus cross force due to cross velocity". Here again, the resultant force may be located either fore or aft, and is replaced by F_{Nu} at the center of gravity and a torque $T_{Nu} = L_{Nu} F_{Nu}$ about the b axis. The torque may have either sign, depending upon whether the center of pressure is fore or aft. T_{Nu} is negative if the center of pressure is forward.

Finally, the Magnus factors due to cross-spin are considered. Here again, the cross spin may be along any axis perpendicular to c. If the cross spin is considered to act about the node line a, as shown in Figure 6, the air velocity relative to the missile will oppose the motion of the air entrained by

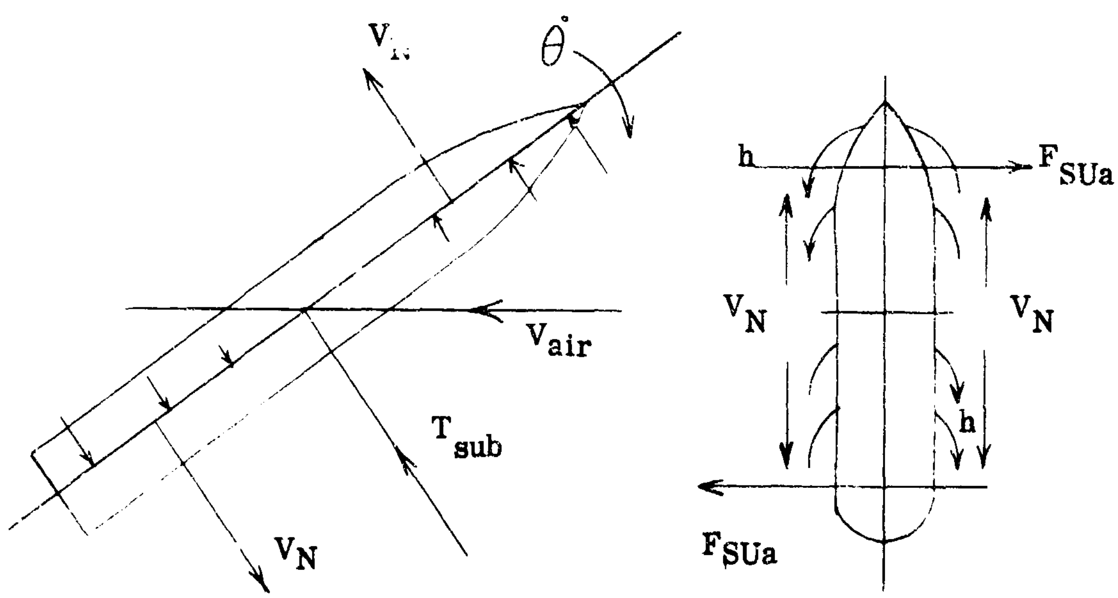

Figure 6

Magnus Factors Due to Cross Spin

the rotation of the missile on the side marked h. On the other side the pressure is lower; hence, there is an unbalanced force toward the right if the center of pressure is fore. The corresponding torque is T_{SUb}, which will have the same sense, whether F_{SUa} acts fore or aft, because F_{SUa} has opposite senses in these two positions.

If the spin is about the b axis, the resulting force is F_{SUb}, and the torque is T_{SUa}. In the general case, the resultants of these two factors will have to be

considered. A "negative" Magnus effect seems to exist for smooth spheres and cylinders at mach numbers below one-half, according to reports from several investigators, the latest from Lyman A. Briggs, Amercian Journal of Physics, 27, 589-96 (1959).

To the above forces and torques must be added the "lag" factors (See BRL 858) due to accelerations in the corresponding originating motions. As these "lag" factors are too small to be measured, they are usually omitted from the equations of motion, except in certain special cases.

Missile Asymmetry

Each missile possesses a slight asymmetry which spins with the (a', b', c) triad. The resultant force may be resolved into axial F_E and normal F_{NE} components. The latter causes the missile to wobble as it spins. The treatment of this eccentricity factor is not completely satisfactory, and cannot be given in general terms because each missile has a different asymmetry. Fortunately, this factor is usually small, and can be treated as having only minor effect.

Jet Factors

The forces and torques exerted by the rocket motor spin within the missile triad (a', b', c), and will be designated by the subscript J.

In an ideal rocket motor the gases ejected from the nozzle exert a perfectly aligned axial force F_{JA} driving the missile forward along the tangent to its path, and there would be no torque. However, in actual rockets this ideal condition does not exist.

In an actual rocket the reactive forces exerted by small unit masses of gas are not constant in time, are not all parallel to the axis, and are not even concurrent. Such a collection of randomly directed forces has for its resultant a force along a "central axis" and a torque about this axis - a combination sometimes called a "wrench". This derivation is known in vector analysis as the "central axis theorem". Owing to the space and time fluctuations of the reactive forces the resultant wrench will fluctuate in magnitude, direction, and point of application. The torque may even change its sense. Fortunately, the fluctuations in gas flow are not very great, and the "average" performance is within practical limits. In general, the central axis of the wrench does not intersect

the rocket axis and is not parallel to it, a condition called malalignment. Rocket engineers usually treat this condition empirically and characterize it by two numbers: "linear" and "angular" malalignment. The linear factor is the distance R_m from the central axis to the center of gravity. The angular factor is derived from the angle B between the central axis and the rocket axis, and is the thrust component normal to the rocket axis. Neither of these names is appropriate as a description of source or effect, because the linear factor is an oblique torque tending to turn and spin the rocket, and the angular factor is a normal force. The effect of these factors is to make the rocket move in an erratic path, erratic because of the fluctuations in gas flow. The torque factor in the wrench is not mentioned in rocket engineering, probably because it is relatively small, and averages out to zero.

Guidance Factors

On jet-driven missiles with zero or very slow spin, there may be devices for steering by means of adjustable fins, lateral jets or some equivalent device. The forces and torques exerted by these devices must also be included in the equations of motion. Whatever the mechanism used, it may produce both axial and normal forces, as well as related torques, and these will be designated as F_{GA} and F_{GN}, etc. They will of necessity rotate with the missile triad (a', b', c).

This completes the catalog of forces and torques acting upon a missile in flight. Their influences on the motion will be reviewed next.

Discussion of Mechanical Effects For Non-Spinning Missile

A nonspinning missile, and the effect of T_N are considered first. If the center of pressure is forward, the missile will overturn. That is, yaw angle Θ will increase and the motion will be unstable, as would be the case with a smooth nonspinning artillery shell. For this reason nonspinning missiles must be stabilized by fins or other surfaces to assure that the center of pressure be aft, and that T_N have a restoring rather than overturning effect. Arrows, javelins, spears are examples of nonspinning missiles having surfaces (e. g., feathers) aft to provide stability.

These additional surfaces provide stability in the form of a torque T_S (or T_M in ballistic nomenclature) which opposes any changes in the yaw angle. This damping torque thus slows down the response of the missile to the restoring moment.

During flight on the upward branch of its trajectory, such a (non-spinning) missile will have little yaw. But at the apex of its path the relative air velocity changes direction, more rapidly with increased steepness of the upward path. This amounts to an increase in the yaw angle, which is partially remedied by the action of T_N, tending to turn the missile axis toward the new direction of the trajectory. Of course, the damping-torque will act to oppose this change, with the net result that the missile will be slowly oriented toward the tangent but will never get quite parallel to it. This is called "mushing in".

There is a possibility that the yaw plane may rotate, owing to some initial disturbance. This produces an angular momentum, which will be conserved, thus causing the continued rotation of the yaw plane. This angular momentum will be changed by the external torques T_N and T_S.

The forces F_N and F_S lie in the yaw plane and produce a lateral acceleration, causing the missile to deviate from its normal trajectory. This deviation will be steadily in one direction if the yaw plane does not rotate. But if this plane rotates, these forces rotate also, causing the missile to follow a helical path.

Historical Note

We come now to the consideration of spinning missiles, but before studying them in detail we might digress for a moment to consider their early history and the part played by Magnus in explaining their behavior. The rifling of small arms to fire a spinning bullet came into general use on the European continent in the 16th Century; the "Kentucky" rifle (made in Pennsylvania) was of this type. History neglects to relate just why or how or by whom the first spinning bullet was fired, but it was soon learned that such a bullet stuck to its line of flight instead of veering from side to side as did a ball fired from a smooth bore.

In early days of artillery cannon balls were used for missiles, but their accuracy was somewhat limited. Benjamin Roberts early in the 18th Century predicted that cannon would some day be rifled to improve their accuracy, the same as small arms; also, he was the first to prove that air currents affected the flight of a cannon ball.

At the beginning of the 19th Century it was noticed that cannon balls did not travel in their expected trajectories, and the discovery that they inadvertently

possessed cross spin was correctly assigned as the cause of their deviations, via the Bernoulli effect. To increase the effectiveness of cannon balls they had to be made bigger, but the increase in gun calibers to shoot them had its limitations, hence cylinders or "long" rounds were introduced. In 1846 Cavelli, in Italy, finally fulfilled Roberts' prediction by making the first rifled cannon, which fired a "long" round. In the 1850's "nearly everyone" experimented with rifled cannon. Most cannon are rifled to produce right-handed spin, although Cranz mentions a left-handed Italian field piece. In 1852, Magnus performed some revealing experiments on air flow past rotating cylinders, and established the existence of the lateral force that was later named after him, the Magnus force. The practice of firing spinning missiles is thus only a hundred years old, and there is still a great deal to be learned about their behavior.

Discussion of Mechanical Effects for Spinning Missile

Returning to the discussion of the motion, we first dispose of the Magnus factors. The Magnus cross force F_{NU} due to cross velocity, and the torque T_{NU} are the ones established by Magnus in 1852. The second set of factors due to cross spin were overlooked by ballisticians for nearly a century, and were only recently recognized by John L. Synge in 1942, when he demonstrated that the force system previously used had been incomplete.

The two forces lie along the node line a, and oppose each other if F_{Su} is aft of the center of gravity. They tend to make the missile follow a spiral path, because the node line rotates, as we shall see later. The two torques lie in the yaw plane, and are in opposition if F_{NU} is aft; being resolved along the oblique axes V and c, they are seen to produce rotation of the yaw plane (precession) and spin about the missile axis.

The effects of the Magnus factors F_{Su} and T_{Su} due to cross spin are hard to measure because they are relatively small; hence, it is permissible to neglect them in most problems.

The other forces F_N and F_S lie in the yaw plane and are in opposition if F_S acts aft. They tend to make the missile travel in a spiral.

The torques T_N and T_S lie along the node line a, and tend to change the yaw angle. T_N is now an overturning torque because F_N acts on the nose of the shell; it is opposed by only T_S, the damping torque, which is in general not great enough to prevent overturning. As noted above, the two Magnus torques

have no effect on the yaw angle and therefore cannot prevent tumbling. The question now arises as to why spinning shells, subject to an overturning torque, have stable motion, i.e., do not tumble. The answer to this question is found in the gyroscopic effects of the spinning shell, which will be treated only qualitatively.

The spin momentum of the shell tends to make it maintain its direction in space. However, the torques T_N and T_S acting on the spin momentum produce a precession about V, i.e., they make the yaw plane rotate. This is the same kind of motion as that executed by a heavy top precessing under the influence of the gravitational torque. Thus, instead of the yaw angle increasing, the shell axis rotates, producing the complex yaw motion.

In common with other missiles, spinning shells are initially well aligned with the tangent and behave fairly well on the upward branch of their trajectory. Trouble begins at the apex of the path, where the tangent changes rather rapidly (depending upon the steepness of the trajectory) from an upward to a downward direction. This amounts to an increase in the yaw angle, and unless a shell is properly designed and spun, it will have unstable motion on the downward branch of the trajectory.

Ballisticians have isolated from the solutions of the equations of motion two characteristics which qualify the performance of a spinning shell: σ the stability and f the tractability. Cranz defines these as $\sigma = C^2 S_c{}^2/4\ AT_N$, which must exceed unity if nutations are to be suppressed, and $f = V_S T_N / C g S_c$, which also must exceed unity if the shell is to remain parallel to the tangent of the path. (V_S is the speed at the apex, and g is the acceleration of gravity.) Since these two factors oppose each other, a compromise must be made in designing a shell that shall have both stability and tractability.

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